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CHANNELS THAT COOPERATIVELY SERVICE A DATA STREAM AND VOICE MESSAGES, II: DIFFUSION APPROXIMATIONS

by

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A system of channels mutually accommodate both data and voice messages, voice having preemptive priority but being a loss system, and data being allowed to queue. Approximations to the data queue properties are derived.

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CHANNELS THAT COOPERATIVELY SERVICE A DATA STREAM AND VOICE MESSAGES, II: DIFFUSION APPROXIMATIONS*

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1. INTRODUCTION

In this paper, the study of the behavior of an element of a communication system carrying both data and voice traffic is continued. In a previous paper, Gaver and Lehoczky (1979), a fluid flow approximation was developed to predict the characteristics of the data queue length for such a system. The fluid flow approach applied to the case in which the data service rate, η, was large compared with the voice service rate, μ; for example, when $\eta/\mu \sim 10^4$. In this paper, a different approach is taken: a Wiener process approximation is developed. The accuracy of this approximation depends on the "heavy traffic" assumption, that is that the overall traffic intensity should approach unity from below. One need not make any assumption concerning the individual parameters. It follows that the diffusion approximation complements the fluid flow approximation to give a more complete picture of the behavior of voice-data communication systems.

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The communication system to be studied is essentially a part of the SENET network as described by Coviello and Vena (1975) or Barbacci and Oakley (1976). This network employs time-slotted frames. A certain portion of each frame is allocated to voice traffic, while any data traffic can use all remaining capacity including any left unused by voice. The voice traffic cannot use any unused data capacity and operates as a loss system.

We introduce probabilistic assumptions, conventional in many queueing studies. Voice traffic arrives according to a Poisson(λ) process, and each voice customer has an independent exponential(μ) service time. Data arrivals are governed by an independent Poisson(δ) process and exhibit independent exponential(η) service times. A total of c channels are reserved for exclusive use of data, while v channels can be used by both data and voice; however, voice pre-empts data. The assumptions imply that voice can be modelled as an M/M/v/v loss system, and the well-known "Erlang B" loss formula will give the loss rate. We focus our attention on the behavior of the data queue and seek to develop expressions for the steady-state distribution and the mean queue length. Many authors have studied such a system, including Halfin and Segal (1972), Halfin (1972), Fischer and Harris (1976), Bhat and Fischer (1976), Fischer (1977), Chang (1977), and Gaver and Lehoczky (1979). Only the latter paper gives approximations valid for the extreme but realistic case in which η/μ is large, and none give a Wiener approximation.

Rather than focus on this simple version of the problem, we wish to provide a somewhat more general analysis. The generalizations arise when one considers more complex voice service. Two types of generalizations are rated here.

In general, voice traffic is unlike data traffic. A voice communication is actually a series of bursts or talkspurts separated by silence. It is reasonable to model a conversation as a two-state Markov chain of alternating talk and silence. Exponential holding times are appropriate for talkspurt and silence lengths, see Brady (1965). During the periods of silence, the voice channel is available for data traffic. This extra channel capacity can serve to allow for increased data utilization, or to reduce the data queue length, or both. The voice process can be described by a Markov chain for which each state provides a description of the number of voice customers requesting service and the number actually using the channel. For each voice state one can compute the number of channels available for data, and thus the data transition rates.

A second generalization allows for the addition of extra voice channels by reducing the quality. One strategy available to provide adequate voice and data capacity is to assign voice users to certain acceptable quality voice channels, say 6 or 8 KBPS. When a specified number of voice users are in the system, new voice users are assigned to less acceptable channels, say with 2 or 4 KBPS. In this way, one has increased the number of voice users that can be serviced at the expense of the quality of the voice transmissions. This situation can also be incorporated into our structure. One again defines voice states: each state provides a description of the voice channels in use. One can specify a generator $\underset{\sim}{Q}$ for the Markov process over the states and use it to compute the channel availability for data for each state. To include this and other possible generalizations, we assume that the voice process is a continuous time Markov chain with finite state space and that it subordinates the data queue length process. We will work out formulas for the special case in which the voice is an M/M/v/v queueing system.

2. Diffusion Approximation Approach

We assume that the voice process $\{V(t),\ t \geq 0\}$ can be described by a Markov chain with state space $\{1,2,\ldots,N\}$. We let this Markov chain have generator $\underset{\sim}{Q} = (q_{ij})$, an $N \times N$ matrix, and stationary distribution $\underset{\sim}{\pi}$. An important special case occurs when voice operates as an M/M/v/v loss system in which case

$$\underset{\sim}{Q} = \mu \underset{\sim}{S} = \mu \begin{pmatrix} -\rho_v & \rho_v & & & \text{\Large 0} \\ 1 & -(1+\rho_v) & \rho_v & & \\ & \ddots & \ddots & \ddots & \\ & & v-1 & -(v-1+\rho_v) & \rho_v \\ \text{\Large 0} & & & v & -v \end{pmatrix} \qquad (2.1)$$

and then $\underset{\sim}{\pi} = (\pi_0, \ldots, \pi_v)$ with

$$\pi_i = \frac{\rho_v^i / i!}{\sum_{j=0}^{v} \rho_v^j / j!}, \qquad \rho_v = \frac{\lambda}{\mu}.$$

Each voice state i gives a number of channels available for data, say c_i. From c_i one can compute the rate, r_i, at which the data queue increases or decreases. Here $r_i = \delta - \eta c_i$. We let

$$\underset{\sim}{R} = \begin{pmatrix} r_1 & & \bigcirc \\ & \ddots & \\ \bigcirc & & r_N \end{pmatrix} \tag{2.2}$$

For stability one requires $\sum_{i=1}^{N} \pi_i r_i < 0$. In the special case for which $\underset{\sim}{Q}$ is given by (2.1), these conditions become

$$\sum_{i=0}^{v} \frac{\rho_v^i}{i!} (\delta - (c + v - i)\eta) < 0$$

where data has exclusive use of c channels and the voice has v channels. This condition can be written $\rho < 1$ where $\rho = (\rho_d + \rho_v(1-q))/(c+v)$, $\rho_d = \delta/\eta$, $\rho_v = \lambda/\mu$, and q is the blocking probability

$$q = \frac{\rho_v^v/v!}{\sum_{j=0}^{v} \rho_v^j/j!}$$

We develop a Wiener process approximation under a heavy-traffic assumption that $\sum_{i=1}^{N} r_i \pi_i \nearrow 0$. This entials finding an appropriate infinitesimal mean and variance. Locally the data queue length process does not resemble a Wiener process. If voice is in state i, the change in the data queue is nearly deterministic and equal to $r_i dt$ rather than a normally distributed random variable. Nevertheless, if we consider a longer period of time, say $[t, t+T]$ with T large, then the change in queue length will be more reasonably assumed to be normally distributed. The next section contains the outline of a proof that a suitably sealed version of the process converges to a Wiener process. This proof does not require the fluid flow approach of η/μ large, so a slightly more refined variance term is derived. The Wiener process derived will have a reflecting barrier at 0.

The voice process subordinates the data process; see Feller [ ], pp. for the notion of subordination. If voice is in state $V(t)$, then the data queue is increasing at rate $r_{V(t)}$ over $[0, t + dt]$. The expected change over $[t, t+T]$ is given by

$$E \int_t^{t+T} r_{V(s)} ds = \int_t^{t+T} E(r_{V(s)}) ds = \int_t^{t+T} \sum_{i=1}^{N} r_i P(v(s)=i) ds .$$

For the case of T large we can assume $V(t)$ has the stationary distribution, thus $V(s)$ has distribution $\underset{\sim}{\pi}$ for all s. The expected change over $[t, t+T]$ thus becomes $T \sum_{i=1}^{N} r_i \pi_i$. Notice that the data queue is very large, so the boundary is not encountered. For a Wiener process, this expected change would be

mT again assuming that the boundary is not encountered. We thus find

$$m = \sum_{i=1}^{N} r_i \pi_i = \underset{\sim}{\pi}^T \underset{\sim}{R} \underset{\sim}{1} \tag{2.3}$$

We follow a similar method to determine σ^2. The variance in the change in the data queue length over $[t, t+T]$ is given by

$$\mathrm{Var} \int_t^{t+T} r_{V(s)} ds \, .$$

We again assume $V(s)$ has distribution $\underset{\sim}{\pi}$, so this variance is equal to

$$\mathrm{Var} \int_0^T r_{V(s)} ds$$

$$= E\left(\int_0^T r_{V(s)} ds\right)^2 - \left(E \int_0^T r_{V(s)} ds\right)^2 = E\left(\int_0^T r_{v(s)}\right)^2 - (\underset{\sim}{\pi} \underset{\sim}{R} \underset{\sim}{1})^2 \, T^2$$

The last term can be computed to be

$$E\left(\int_0^T r_{V(s)} ds\right)^2 = E \int_0^T \int_0^T r_{V(s)} r_{V(t)} dsdt = \int_0^T \int_0^T E(r_{V(s)} r_{V(t)}) dsdt$$

$$= 2 \int_0^T \int_0^T \sum_{i=1}^{N} r_i \, P(V(s)=i) \, E(r_{V(t)} | V(s)=i) dsdt$$

$$= 2 \int_0^T \int_0^T \sum_{i=1}^{N} r_i \pi_i \sum_{j=1}^{N} r_j \, P(v(t)=j | V(s)=i) dsdt \, .$$

We let $p_{ij}(u) = P(V(u)=j \mid V(0)=i)$ and $\underset{\sim}{P}(u) = (p_{ij}(u))$. The Kolmogorov forward equations give $\underset{\sim}{P}(u) = \exp(\underset{\sim}{Q}u)$ where $\exp(\underset{\sim}{M}) = \underset{\sim}{I} + \underset{\sim}{M} + \underset{\sim}{M}^2/2! + \cdots$ for a square matrix $\underset{\sim}{M}$. One can reexpress the second moment in matrix form as

$$E\left(\int_0^T r_{V(s)} ds\right)^2 = 2 \int_0^T \int_0^t \underset{\sim}{\pi}\underset{\sim}{R} \exp(\underset{\sim}{Q}(t-s)) \underset{\sim}{R}\underset{\sim}{1} \, dsdt$$

$$= 2\underset{\sim}{\pi}\underset{\sim}{R} \int_0^T \int_0^t \exp(\underset{\sim}{Q}u) \, dudt \, \underset{\sim}{R}\underset{\sim}{1} .$$

In view of the fact that $\underset{\sim}{Q}$ is singular, one can most easily carry out the required integration by introducing the eigenvalue decomposition $\underset{\sim}{Q} = \underset{\sim}{\Phi}\underset{\sim}{D}\underset{\sim}{\Psi}$ with

$$\underset{\sim}{D} = \begin{pmatrix} \theta_1 & & & \bigcirc \\ & \theta_2 & & \\ & & \ddots & \\ \bigcirc & & & \theta_N \end{pmatrix}$$

a diagonal matrix of eigenvalues,

$$\underset{\sim}{\Phi} = (\underset{\sim}{1}, \underset{\sim}{\phi}_2, \ldots , \underset{\sim}{\phi}_N), \qquad \underset{\sim}{\Psi} = (\underset{\sim}{\pi}, \underset{\sim}{\psi}_2, \ldots , \underset{\sim}{\psi}_N)^T ,$$

the associated right- and left-eigenvectors. Here $\underset{\sim}{\Phi}\underset{\sim}{\Psi} = \underset{\sim}{\Psi}\underset{\sim}{\Phi} = I$. Since $\underset{\sim}{Q}$ is irreducible and finite θ_i has a negative real part for $2 \leq i \leq N$. It follows that

$$\int_0^T \int_0^t \exp(\underset{\sim}{Q}u)\,dudt = \underset{\sim}{\Phi} \int_0^T \int_0^t \exp(\underset{\sim}{D}u)\,dudt$$

$$\underset{\sim}{\Psi} = 2\underset{\sim}{\Phi} \begin{pmatrix} T^2/2 & & & \bigcirc \\ & \alpha_2 & & \\ & & \ddots & \\ \bigcirc & & & \alpha_N \end{pmatrix} \underset{\sim}{\Psi}$$

with $\alpha_i = \theta^{-i}(\theta^{-i}(e^{\theta_i T} - 1))$. Since θ_i has a negative real part $\alpha_i \sim -T/\theta_i$ as $T \to \infty$. For large T, the second moment can be written approximately as

$$2\underset{\sim}{\pi}^T \underset{\approx}{R}\underset{\approx}{\Phi} \begin{pmatrix} T^2/2 & & & \bigcirc \\ & -T\theta_1^{-1} & & \\ & & \ddots & \\ \bigcirc & & & -T\theta_N^{-1} \end{pmatrix} \underset{\approx}{\Psi}\underset{\approx}{R}\underset{\approx}{1} ,$$

thus the variance is given by

$$T^2 \underset{\sim}{\pi}^+ \underset{\sim}{R} \begin{pmatrix} 1 & & & \bigcirc \\ & 0 & & \\ & & \ddots & \\ \bigcirc & & & 0 \end{pmatrix} \underset{\approx}{\Psi}\underset{\approx}{R}\underset{\approx}{1} - 2T\underset{\sim}{\pi}^T \underset{\approx}{R}\underset{\approx}{\Phi} \begin{pmatrix} 0 & & & \bigcirc \\ & \theta_2^{-1} & & \\ & & \ddots & \\ \bigcirc & & & \theta_N^{-1} \end{pmatrix} \underset{\approx}{\Psi}\underset{\approx}{R}\underset{\approx}{1}$$

$$- T^2 (\underset{\sim}{\pi}^+ \underset{\approx}{R}\underset{\approx}{1})^2 .$$

The first and last terms cancel to give

$$\mathrm{Var}\left(\int_0^T r_{V(s)}\,ds\right) = -2T\pi^T R\Phi \begin{pmatrix} 0 & & & \bigcirc \\ & \theta_2^{-1} & & \\ & & \ddots & \\ \bigcirc & & & \theta_N^{-1} \end{pmatrix} \Psi R1 \tag{2.4}$$

For a Wiener process, this variance would be given by $T\sigma^2$. We let

$$Q^- = \Phi \begin{pmatrix} 0 & & & \bigcirc \\ & \theta_2^{-1} & & \\ & & \ddots & \\ \bigcirc & & & \theta_N^{-1} \end{pmatrix} \Psi \tag{2.5}$$

so Q^- is a reflexive generalized inverse of Q obtained by reciprocating nonzero eigenvalues. We note that both $QQ^-Q = Q$ and $Q^-QQ^- = Q^-$ hold, thus we write

$$\begin{aligned} \frac{\sigma^2}{2} &= -\pi^T RQ^- R1 \\ m &= \pi^+ R1 < 0 \; . \end{aligned} \tag{2.6}$$

The crucial parameter $2m/\sigma^2$ can thus be calculated.

We illustrate the formulas in (2.6) for the special case where Q is given by (2.1). For this case

$$\frac{\sigma^2}{2} = \frac{1}{\mu} \underset{\sim}{\pi}^T \underset{\approx}{R}\underset{\approx}{\Phi} \begin{pmatrix} 0 & & & \bigcirc \\ & \theta_1^{-1} & & \\ & & \ddots & \\ \bigcirc & & & \theta_v^{-1} \end{pmatrix} \underset{\approx}{\Psi}\underset{\approx}{R}\underset{\sim}{1} = \frac{1}{\mu} \underset{\sim}{\pi}^T \underset{\approx}{R}\underset{\approx}{S}^- \underset{\approx}{R}\underset{\sim}{1}$$

$$= \frac{1}{\mu} \underset{\sim}{\pi}^T \underset{\approx}{R}\underset{\approx}{S}^- (\underset{\approx}{R} - m\underset{\approx}{I})\underset{\sim}{1} + \frac{m}{\mu} \underset{\sim}{\pi}^T \underset{\approx}{R}\underset{\approx}{S}^- \underset{\sim}{1} \ .$$

For our choice of $\underset{\approx}{S}^-$, $\underset{\approx}{S}^-\underset{\sim}{1} = \underset{\sim}{0}$, thus

$$\frac{\sigma^2}{2} = \frac{1}{\mu} \underset{\sim}{\pi}^T \underset{\approx}{R}\underset{\approx}{S}^- (\underset{\approx}{R} - m\underset{\approx}{I})\underset{\sim}{1} = \frac{1}{\mu} \underset{\sim}{\pi}^T \underset{\approx}{R}\underset{\approx}{S}^- \underset{\sim}{y}$$

where $\underset{\sim}{y} = (\underset{\approx}{R} - m\underset{\approx}{I})\underset{\sim}{1}$. We note that $\underset{\sim}{\pi}^T\underset{\sim}{y} = 0$, so if $\underset{\sim}{z} = \underset{\approx}{S}^-\underset{\sim}{y}$, then $\underset{\approx}{S}\underset{\sim}{z} = \underset{\sim}{y}$ is a consistent set of equations and $\underset{\sim}{\pi}^T\underset{\sim}{z} = 0$.

In general one can solve (2.6) by direct calculation of $\underset{\approx}{Q}^-$; however, this is rarely possible. A second approach is to solve $\underset{\approx}{Q}\underset{\sim}{z} = \underset{\sim}{y}$ subject to $\underset{\sim}{\pi}^T\underset{\sim}{z} = \underset{\sim}{0}$. For $\underset{\approx}{Q}$ of form (2.1), this can be carried out. The equations become

$$\begin{aligned} \rho_v(z_i - z_{i-1}) - (i-1)(z_{i-1} - z_{i-2}) &= r_{i-1} - m \ , \ 1 \leq i \leq v \\ -v(z_v - z_{v-1}) &= r_n - m \\ \sum_{i=0}^{v} \frac{\rho_v^i}{i!} z_i &= 0. \end{aligned} \tag{2.7}$$

One can easily find

$$z_k - z_{k-1} = \frac{\sum_{j=0}^{k-1} \pi_j (r_j - m)}{\rho_v \pi_{k-1}} .$$

We define

$$T_k = \sum_{j=0}^{k} \pi_j (r_j - m).$$

Thus $T_v = 0$ and $z_k - z_{k-1} = T_{k-1}/\rho_v \pi_{k-1}$. Moreover

$$z_k = z_0 + \sum_{j=1}^{k} z_j = z_0 + \sum_{j=1}^{k} \frac{T_{j-1}}{\rho_v \pi_{j-1}} .$$

The $\sum_{i=0}^{v} \pi_i z_i = 0$ condition gives

$$z_0 = - \sum_{k=1}^{v} \pi_k \sum_{j=1}^{k} \frac{T_{j-1}}{\rho_v \pi_{j-1}} .$$

The vector $\underset{\sim}{z}$ has now been explicitly determined, and it remains to compute

$$\frac{1}{\mu} \underset{\sim}{\pi}^T \underset{\sim}{R} \underset{\sim}{z} = \frac{1}{\mu} \underset{\sim}{\pi}^T \underset{\sim}{R} (z_0 \underset{\sim}{1} + (0, u_1, \ldots , u_v))$$

with

$$u_k = \sum_{j=1}^{k} \frac{T_{j-1}}{\rho_v T_{j-1}} .$$

We find

$$\frac{1}{\mu}\,\pi^T Rz = -\frac{m}{\mu}\,z_0 + \frac{1}{\mu}\sum_{k=1}^{v}\pi_k r_k \sum_{i=1}^{k}\frac{T_{i-1}}{\rho_v\pi_{i-1}}$$

$$= \frac{1}{\mu}\sum_{k=1}^{v}\pi_k(r_k-m)\sum_{i=1}^{k}\frac{T_{i-1}}{\rho_v\pi_{i=1}}$$

$$= \frac{1}{\mu}\sum_{i=1}^{v}\frac{T_{i-1}}{\rho_v\pi_{i-1}}\sum_{k=i}^{v}\pi_k(r_k-m) = \frac{1}{\mu}\sum_{i=1}^{v}\frac{T_{i-1}}{\rho_v\pi_{i-1}}\left(-\sum_{k=0}^{i-1}\pi_k(r_k-m)\right)$$

$$= -\frac{1}{\mu}\sum_{i=1}^{v}\frac{T_{i-1}^2}{\rho_v\pi_{i-1}} = -\frac{1}{\mu}\sum_{i=0}^{v-1}\frac{T_i^2}{\rho_v\pi_i}\,.$$

We thus derive

$$\frac{\sigma^2}{2} = \frac{1}{\mu}\sum_{i=0}^{v-1}\frac{T_i^2}{v\ i}\,, \qquad T_i = \sum_{j=0}^{i}\pi_j(r_j-m) \tag{2.8}$$

For the special case considered $r_j = S - (c+v-j)\eta = \eta(\rho_d - (c+v) + j)$. One finds

$$m = \sum_{i=0}^{v}\eta(\rho_d - (c+v)+j)\,\frac{\rho_v^j}{j!S_v}$$

$$= \eta[\rho_d - (c+v) + \rho_v(1-q)]$$

where $q = \rho_v^v/v!S_v$ is the blocking probability. Thus $m = -\eta(c+v)(1-\rho)$. Plugging into (2.8) we find

$$\frac{\sigma^2}{2} = \frac{\eta^2}{\rho_v \mu} \sum_{i=0}^{v-1} \frac{\left(\sum_{j=0}^{i} \frac{\rho_v^j}{j!} \left[\rho_d - (c+v)\rho + j\right]\right)^2}{S_v \rho_v^i / i!} .$$

This can be simplified by letting $q_i = \rho_v^i / i! S_i$, the blocking probability for an M/M/i/i system with traffic intensity ρ_v. We find

$$\frac{\sigma^2}{2} = \frac{\eta^2 \rho_v}{\mu} \sum_{i=0}^{v-1} \frac{S_i}{S_v} \frac{(q_i - q_v)^2}{q_i} , \qquad S_i = \sum_{j=0}^{i} \frac{\rho_v^j}{j!} . \tag{2.9}$$

The special case $v = 1$ gives

$$\frac{\sigma^2}{2} = \frac{\eta^2}{\mu} \frac{\rho_v}{(1 + \rho_v)^2} .$$

We thus have given explicit formulas for the infinitesimal mean and variance of a Wiener process with reflecting barrier at 0. One can therefore use standard results for Wiener processes with reflecting barriers to derive the stationary distribution--it will be exponential with parameter $2m/\sigma^2$. The stationary mean queue length will thus be $\sigma^2/2m$; where σ^2 and m are given by (2.6) and (2.8). For example when $v = 1$ and $c = 0$

$$\frac{\sigma^2}{2m} = \frac{\eta}{\mu} \frac{\rho_v}{(1-\rho)(1+\rho_v)^3} .$$

This is developed under heavy traffic conditions, i.e., $\rho \nearrow 1$. For such conditions

$$\rho_d + \frac{\rho_v}{1 + \rho_v} \to 1$$

or

$$\left| \rho_d - \frac{1}{1 + \rho_v} \right| \to 0 \quad .$$

It follows that the above expression is also

$$\frac{\eta}{\mu} \frac{\rho_v \rho_d}{(1-\rho)(1+\rho_v)^2}$$

in agreement with Fischer (1977) and Gaver and Lehoczky (1979).

The characterization of the data queue as a Wiener process with reflecting barrier allows one to discuss the dynamics of the data queue. For example, suppose the data queue is at level x at time t where x is large. One might wish to study the time that elapses until the queue becomes empty. This is the duration of a busy period started at x, and corresponds to a first-passage time for a Wiener process which we denote by t_x. Moreover one might also be interested in the area beneath the sample path until it reaches the boundary of Q, since this area gives the total time waited by all customers involved in the busy period. We let this area be $A(x)$ and seek to compute $t_x = E(T_x)$ and $a(x) = E(A(x))$. Straightforward backward equation or martingale arguments give

$$t_x = -x/m$$

$$a(x) = -\frac{x^2}{2m} + \frac{\sigma^2}{2m^2}\, x \tag{2.10}$$

where m is negative, and m and σ^2 are given earlier. The distribution of the first passage time can also be easily calculated; however, its Laplace transform is most convenient to find and is given by

$$E(e^{-s\tau_x}) = \exp\left[\left(\frac{x}{\sigma}\right) \quad -\left(\frac{m}{\sigma}\right) - \sqrt{\left(\frac{m}{\sigma}\right)^2 + 2s} \right] \tag{2.11}$$

4. A Refined Diffusion Approximation

In this section we continue the development of a diffusion approximation for the data queue length process. The analysis is patterned after the important semigroup methods of Burman [1979]. One develops a sequence of Markov processes and studies the behavior of their generators. One shows that the generators converge and hence concludes that the associated semi-groups converge. This entails the convergence of the finite dimensional distributions. The limiting finite dimensional distributions will be those of a certain Brownian motion. The theory underlying this approach is based on theorems of Trotter and Kato, Kurtz and Burman; the reader is referred to Burman (1979) for technical details. We will illustrate the approach in the context of the special case $v = 1$, $c = 0$. The method applies to the case of general c and v but the details are not given here.

Let $\{V(t), t \geq 0\}$ be the voice process, a Markov process with state space $\{0,1\}$. Let $N(t)$ be the data system size at time t. We study the sequence of Markov processes $\langle\{(X_n(t), V_n(t)), t \geq 0\}\rangle_{n=1}^{\infty}$ where

$$X_n(t) = N(nt)/\sqrt{n} \quad \text{and} \quad V_n(t) = V(nt) \ . \tag{3.1}$$

The generator of this bivariate process is easy to compute since $V_n(t)$ subordinates $X_n(t)$. Let $f(x,k)$ be a function with domain $[0,\infty) \times \{0,1\}$ which is smooth as a function of x for each $k = 0,1$. The generator is given by

$$A_n f(x,k) = \begin{cases} \delta n[f(x + 1/\sqrt{n},\ k) - f(x,k)] \\ \quad + (1-k)\eta n[f(x - 1/\sqrt{n},k) - f(x,k)] + nQf(x,k) \\ \qquad\qquad\qquad \text{for} \quad x \geq 1/\sqrt{n} \\ \\ \delta n[f(1/\sqrt{n},\ k) - f(0,k)] + nQf(0,k) \\ \qquad\qquad\qquad \text{for} \quad x = 0\ . \end{cases} \tag{3.2}$$

where Q is the generator of the M/M/1/1 voice process and is given by (2.1).

Next expand in a Taylor series and collect terms to rewrite (3.2) as

$$A_n f(x,k) = \begin{cases} nQf(x,k) + \sqrt{n}\ f_x(x,k)(\delta - (1-k)\eta) \\ \qquad + \frac{1}{2} f_{xx}(x,k)(\delta + (1-k)\eta) + o(1) \\ \qquad\qquad\qquad \text{for} \quad x \geq 1/\sqrt{n} \\ \\ nQf(0,k) + \sqrt{n}\ f_x(0,k)\delta + \frac{1}{2} f_{xx}(x,k)\delta + o(1) \\ \qquad\qquad\qquad \text{for} \quad x = 0 \end{cases} \tag{3.3}$$

where $f_x(x,k) = \frac{\partial}{\partial x} f(x,k)$ and $f_{xx}(x,k) = \frac{\partial^2}{\partial x} f(x,k)$.

We wish to let $n \to \infty$ and focus attention on the data queue process alone. To derive a diffusion limit for the data queue process alone, we introduce a sequence of functions $\langle f_n(x,k)\rangle_{n=1}^{\infty}$ for which $f_n(x,k) \to f(x)$, where f is a smooth function satisfying $f'(0) = 0$. We wish to study the limiting behavior of $A_n f_n(x,k)$. Let

$$f_n(x,k) = f(x) + \frac{1}{\sqrt{n}}\, g(x,k) + \frac{1}{n}\, h(x,k)$$

where g and h are smooth. Clearly $f_n(x,k) \to f(x)$. Substitution in (3.3) gives

$$A_n f_n(x,k) = \begin{cases} nQf(x) + \sqrt{n}\,[Qg(x,k) + f'(x)(\delta - (1-k)\eta)] \\ \quad + [Qh(x,k) + g_x(x,k)(\delta - (1-k)\eta) + \frac{1}{2} f''(x)(\delta + (1-k)\eta)] \\ \qquad\qquad\qquad\qquad x \geq 1/\sqrt{n} \\ nQf(0) + \sqrt{n}\,[Qg(0,k) + f'(0)\delta] \\ \quad + [Qh(0,k) + g_x(0,k)\delta + \frac{1}{2} f''(0)(\delta)] \qquad x = 0 \, . \end{cases} \tag{3.4}$$

We first note that $Qf(x) = 0$ since Q operates on the voice or k component only, not on the data. This eliminates the first terms $nQf(x)$ and $nQf(0)$. We next examine the $Qg(x,k) + f'(x)(\delta - (1-k)\eta)$ term. This can be rewritten as

$$Qg(x,k) + f'(x)[\delta - (1-k)\eta - ((\delta-\eta) + \delta\rho_v)/(1+\rho_v)$$

$$+ f'(x)((\delta-\eta) + \delta\rho_v)/(1+\rho_v)$$

$$= Qg(x,k) + f'(x)\left[-\frac{\rho_v}{1+\rho_v} + k\right] = f'(x)\ \eta(1-\rho) \ .$$

We now select the function $g(x,k)$ so that

$$Qg(x,k) = -f'(x)\left[-\frac{\rho_v}{1+\rho_v} + k\right]\eta \ .$$

The function $g(x,k)$ must satisfy the equations

$$\begin{aligned} \lambda(g(x,1) - g(x,0)) &= f'(x)\,\frac{\eta\rho_v}{1+\rho_v} \\ -\mu(g(x,1) - g(x,0)) &= -f'(x)\,\frac{\eta}{1+\rho_v} \end{aligned} \tag{3.5}$$

The above equations are consistent and redundant, thus any $g(x,k)$ for which

$$g(x,1) - g(x,0) = \frac{f'(x)\eta}{\mu(1+\rho_v)}$$

will suffice. We select

$$\begin{aligned} g(x,0) &= \frac{1}{2}\,f'(x) \\ g(x,1) &= \left(\frac{1}{2} + \frac{\eta}{\mu(1+\rho_v)}\right)\,f'(x) \end{aligned} \tag{3.6}$$

or

$$g(x,k) + \left(\frac{1}{2} + \frac{\eta k}{\mu(1+\rho_v)}\right)\,f'(x) \ .$$

This choice of g gives

$$A_n f_n(x,k) = \begin{cases} -\sqrt{n}\,\eta(1-\rho)\ f'(x) + [Qh(x,k) + \left(\frac{1}{2} + \frac{\eta k}{\mu(1+\rho_v)}\right) f''(x)(\delta - (1-k)\eta) \\ \quad + \frac{1}{2} f''(x)(\delta + (1-k)\eta)], \qquad x \geq 1/\sqrt{n} \\ \sqrt{n}\left(f'(0)\delta + f'(0)\ \frac{\eta(\rho_v - k(1+\rho_v))}{\mu(1+\rho_v)}\right) \\ \quad + \left[Qh(0,k) + \left(\frac{1}{2} + \frac{\eta k}{\mu(1+\rho_v)}\right) f''(0)\delta + \frac{1}{2} f''(0)\delta\right] \\ \qquad x = 0 \end{cases} \tag{3.7}$$

Equation (3.7) can be rewritten recalling $f'(0) = 0$ as

$$A_n f_n(x,k) = \begin{cases} -\sqrt{n}\,\eta(1-\rho) f'(x) + \left[Qh(x,k) + \eta f''(x) \left\{ \frac{1}{2} + \frac{\eta k}{\mu(1+\rho_v)}\ (\rho_d - (1-k)) \right.\right. \\ \quad \left.\left. + \frac{1}{2}(\rho_d + (1-k)) \right\} \right] + o(1), \quad x \geq 1/\sqrt{n} \\ Qh(0,k) + \eta f''(0) \left[\rho_d\ \frac{1}{2} + \frac{\eta k}{\mu(1+\rho_v)} + \frac{1}{2}\rho_d\right], \ x = 0 \end{cases} \tag{3.8}$$

or

$$A_n f_n(x,k) = \begin{cases} -\sqrt{n}\,\eta(1-\rho)\ f'(x) + \left[Qh(x,k) + f''(x)\eta\cdot\rho_d\left(1 + \frac{\eta k}{\mu(1+\rho_v)}\right)\right] + o(1) \\ \qquad x \geq 1/\sqrt{n} \\ Qh(0,k) + f''(0)\ \eta\rho_d\left(1 + \frac{\eta k}{\mu(1+\rho_v)}\right) + o(1), \quad x = 0 \end{cases} \tag{3.9}$$

We rewrite

$$Qh(x,k) + f''(x)\eta\rho_d \quad 1 + \left(\frac{\eta_k}{\mu(1+\rho_v)}\right)$$

as

$$\phi h(x,k) + f'(x)\,\frac{\eta^2\rho_d}{\mu(1+\rho_v)}\left(k - \frac{\rho_v}{1+\rho_v}\right) + f''(x)\eta\rho_d\left(\frac{\eta\rho_v}{\mu(1+\rho_v)^2} + 1\right)$$

and choose $h(x,k)$ so that

$$Qh(x,k) = -f''(x)\,\frac{\eta^2\rho_d}{\mu(1+\rho_v)}\left(k - \frac{\rho_v}{1+\rho_v}\right)$$

The function $h(x,k)$ must satisfy

$$\lambda(h(x,1) - h(x,0)) = f''(x)\,\frac{\eta^2\rho_d\rho_v}{\mu(1+\rho_v)^2}$$

$$-\mu(h(x,1) - h(x,0)) = -\,f''(x)\,\frac{\eta^2\rho_d}{\mu(1+\rho_v)^2} \tag{3.10}$$

These equations are consistent and redundant so a manifold of solutions are possible. Any one will suffice. This choice of $h(x,k)$ allows one to rewrite (3.9) as

$$A_n f_n(x,k) = \begin{cases} -\sqrt{n}\,\eta(1-\rho)\, f'(x) + f''(x)\,\eta\rho_d\left(\dfrac{\eta\rho_v}{\mu(1+\rho_v)^2} + 1\right) + o(1) \\ \qquad\qquad\qquad\qquad x \geq 1/\sqrt{n} \\ f''(0)\,\eta\rho_d\left(\dfrac{\eta\rho_v}{\mu(1+\rho_v)^2} + 1\right) + o(1) \end{cases} \tag{3.11}$$

We now let $n \to \infty$. In order for (3.11) to converge to a sensible limit, we must invoke the heavy traffic approximation, that $\rho \to 1$. Specifically we let $\rho = \rho_n = 1 - (\theta/\sqrt{n})$ for some $\theta > 0$. Equation (3.11) becomes in the limit

$$A_n f_n(x,k) \longrightarrow$$

$$Af(x) = \begin{cases} -\eta\theta f'(x) + f''(x)\eta\rho_d\left(\dfrac{\eta\rho_v}{\mu(1+\rho_v)^2} + 1\right), & x > 0 \\ f''(0)\eta\rho_d\left(\dfrac{\eta\rho_v}{\mu(1+\rho_v)^2} + 1\right), & x = 0 \\ f'(0) = 0 \end{cases} \tag{3.12}$$

This is the generator of a Wiener process with reflecting barrier at 0, drift of $-\eta\theta$, and

$$\frac{\sigma^2}{2} = \eta\rho_d\left(\frac{\eta\rho_v}{\mu(1+\rho_v)^2} + 1\right) .$$

The stationary distribution for such a process is an exponential distribution with parameter

$$\theta \Big/ \left\{ \rho_d \left(\frac{\eta \rho_v}{\mu(1+\rho_v)^2} + 1 \right) \right\} ,$$

thus the mean queue length is given by

$$\frac{\rho_d}{\theta} \left(\frac{\eta \rho_v}{\mu(1+\rho_v)^2} + 1 \right) .$$

It is informative to try to apply this heavy traffic result to a case in which ρ is near but less than 1. One might replace θ by $\sqrt{n}\,(1-\rho)$. Since we have scaled by $\sqrt{n}$, the stationary distribution of $N(t)$, the unscaled queue length should be approximately exponential with parameter

$$\frac{(1-\rho)}{\rho_d \left(\dfrac{\eta \rho_v}{\mu(1+\rho_v)^2} + 1 \right)}$$

The mean queue length becomes

$$\frac{\rho_d}{1-\rho} \left(\frac{\eta \rho_v}{\mu(1+\rho_v)^2} + 1 \right) .$$

This result is in exact agreement with Fischer (1977) and is therefore exact. The fluid flow approximation treats η/μ as being large, hence $[\eta\rho_v/\mu(1+\rho_v)^2]$ is assumed to dominate 1. This gives $(\eta\, \rho_d \rho_v)/[\mu(1+\rho_v)^2(1-\rho)]$ as the fluid flow mean queue length found in Gaver and Lehoczky (1979) and in Section 2 of this paper.

The diffusion approximation thus found represents a refinement of the fluid flow diffusion approximation. Even if η/μ is large, ρ_V may be small, so the 1 may be important.

Once the Wiener process infinitesimal drift and variance have been found, one can also use the dyanmics of the Wiener process to model the dynamics of the data queue process. Busy period distributions, areas, and transient behavior in general can be determined.

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